MATH4240: Stochastic Processes Tutorial 6

WONG, Wing Hong

The Chinese University of Hong Kong

whwong@math.cuhk.edu.hk

15 March, 2021

Uniqueness of Stationary Distribution

Theorem 1. Let P be a stochastic $n \times n$ matrix over a finite state space $\mathcal{S}$. If P satisfies the following assumptions:

Assumption 1. The left eigenvector w.r.t. 1 can be chosen to have all nonnegative entries.

Assumption 2. The eigenvalue 1 is a simple root of the characteristic polynomial of P.

Assumption 3. Except 1, all other eigenvalues have moduli less than 1.

Then, the chain has a unique stationary distribution π and

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix}$$

Uniqueness of Stationary Distribution

Theorem 2. An irreducible positive-recurrent ($m_x < \infty$) Markov chain has a unique stationary distribution π, given by

$$\pi(x) = \frac{1}{m_x},$$

where $x \in \mathcal{S}$ and $m_x = E_x(T_x) < \infty$.
In particular, every irreducible finite Markov chain has a unique stationary distribution.

Example 1 : $\mathcal{S} = \{0, 1, 2, \ldots\}$,

$$P(x, x+1) = p \quad \text{and} \quad P(x, 0) = 1 - p, \quad 0 < p < 1.$$

Suppose that the stationary distribution π exists. Then $\pi P = \pi$ and $\sum_{x=0}^{\infty} \pi(x) = 1$ imply that

$$\begin{aligned}
\pi(0) &= \sum_{x=0}^{\infty} \pi(x) P(x, 0) = (1-p) \sum_{x=0}^{\infty} \pi(x) = 1 - p, \\
\pi(1) &= \pi(0) P(0, 1) = (1-p)p, \\
\pi(2) &= \pi(1) P(1, 2) = (1-p)p^2, \\
&\ldots
\end{aligned}$$

By induction, $\pi(x) = (1-p)p^x$, $x \geq 0$.

Examples on Stationary Distribution

On the other hand, we check directly that above π satisfies both $\sum_{x=0}^{\infty} \pi(x) = 1$ and $\pi(x) = \sum_{y=0}^{\infty} \pi(y)P(y,x)$, $x \geq 0$. Hence $\pi = (1-p, (1-p)p, (1-p)p^2, \cdots)$ is the unique (why unique?) stationary distribution.

Examples on Stationary Distribution

Example 2: $\mathcal{S} = \{1, 2, \ldots, d\}$, $d < \infty$. The transition function P is *doubly stochastic*, that is to say,

$$\sum_{x \in \mathcal{S}} P(x, y) = 1.$$

Assume the chain is irreducible. Since all states are in a finite irreducible closed set, the stationary distribution is unique.

Let $\pi(x) = \frac{1}{d}$ for all $x \in \mathcal{S}$. Then it is a probability vector since $\sum_{x=1}^{d} \pi(x) = 1$. Moreover, for all $y \in \mathcal{S}$,

$$\sum_{x=1}^{d} \pi(x) P(x, y) = \sum_{x=1}^{d} \frac{1}{d} P(x, y) = \frac{1}{d} = \pi(y).$$

This shows π is the unique stationary distribution we want.

Examples on Stationary Distribution

Example 3: $\mathcal{S} = A \cup B$, where $A = \{1, 2, \ldots, c\}$ and $B = \{c+1, c+2, \ldots, c+d\}$. The transition probability

$$P(x, y) = \begin{cases} 1/d, & x \in A \text{ and } y \in B, \\ 1/c, & x \in B \text{ and } y \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Since the chain is irreducible and finite, it has a unique stationary distribution π.

For $y \in A$, we have

$$\pi(y) = (\pi P)(y) = \sum_{x \in B} \pi(x) P(x, y) = \frac{1}{c} \sum_{x \in B} \pi(x),$$

which implies

$$\sum_{y \in A} \pi(y) = \sum_{y \in A} \frac{1}{c} \sum_{x \in B} \pi(x) = \sum_{x \in B} \pi(x).$$

Note that $\sum_{x \in A \cup B} \pi(x) = 1$. Hence $\sum_{y \in A} \pi(y) = \sum_{x \in B} \pi(x) = 1/2$. Thus for any $y \in A$, $\pi(y) = \frac{1}{2c}$.

Examples on Stationary Distribution

For $z \in B$, we have

$$\pi(z) = (\pi P)(z) = \sum_{x \in A} \pi(x) P(x, z) = \frac{1}{d} \sum_{x \in A} \pi(x) = \frac{1}{2d}.$$

Therefore, the stationary distribution is

$$\pi(x) = \begin{cases} \frac{1}{2c}, & x \in A, \\ \frac{1}{2d}, & x \in B. \end{cases}$$

Examples on Stationary Distribution

Example 4. $\mathcal{S} = \{1, 2, 3, 4\}$,

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ P = & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{array}$$

Observe that there are two '3-circles': $1 \to 2 \to 3 \to 1$ and $2 \to 3 \to 4 \to 2$. Hence any two states can communicate, so the chain is irreducible.

Examples on Stationary Distribution

Let $\pi = (\pi(1), \pi(2), \pi(3), \pi(4))$ be the stationary distribution. Then $\pi P = \pi$ implies that

$$\begin{cases} (2/3)\pi(3) = \pi(1), \\ \pi(1) + \pi(4) = \pi(2), \\ \pi(2) = \pi(3), \\ (1/3)\pi(3) = \pi(4). \end{cases}$$

Together with $\pi(1) + \pi(2) + \pi(3) + \pi(4) = 1$, we get $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) = (2/9, 1/3, 1/3, 1/9)$.

Example 5. note that

$$P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \end{pmatrix},$$

$$P^4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \end{pmatrix} = P.$$

Inductively we have $P^{3k-2} = P$, $P^{3k-1} = P^2$, and $P^{3k} = P^3$ for all $k \geq 1$. Hence the limit $\lim_{k\to\infty} P^k$ does not exists even the chain has the unique stationary distribution.

Examples on Stationary Distribution

Example 6. Let a random walk defined on integer with

$$P(i, i+1) = P(i, i-1) = \frac{1}{2}$$

for all $i \in \mathbb{Z}$. It is an irreducible Markov chain by tutorial 4, but it does not have any stationary distributions.

Suppose not. Let π be a stationary distribution. Since $\sum_{x \in \mathcal{S}} \pi(x) = 1$, there exists some $i \in \mathbb{Z}$ such that $\pi(i) > 0$. Since

$$\pi(i) = (\pi P)(i) = \sum_{x \in \mathbb{Z}} \pi(x) P(x, i) = \frac{1}{2}\pi(i-1) + \frac{1}{2}\pi(i+1),$$

Then, we have either $\pi(i+1) \geq \pi(i)$ or $\pi(i-1) \geq \pi(i)$. Without loss of generality, suppose $\pi(i+1) \geq \pi(i)$.

Examples on Stationary Distribution

If $\pi(i+1) = \pi(i)$, since $\pi P = \pi$, we have $\pi(i+1) = \frac{1}{2}\pi(i) + \frac{1}{2}\pi(i+2)$. Then, $\pi(i+2) = \pi(i+1) = \pi(1)$. By induction, we have $\pi(i+k) = \pi(i) > 0$ for all $k \in \mathbb{N}$. In particular,

$$\sum_{x\in\mathcal{S}} \pi(x) \geq \sum_{k=0}^{\infty} \pi(i+k) = \sum_{k=0}^{\infty} \pi(i) = \infty,$$

which is a contradiction.

Suppose $\pi(i+1) > \pi(i)$, then by similar argument, we have $\pi(i+2) > \pi(i+1) > \pi(i)$. Again, we have

$$\sum_{x\in\mathbb{Z}} \pi(x) \geq \sum_{k=0}^{\infty} \pi(i+k) > \sum_{k=0}^{\infty} \pi(i) = \infty,$$

which is again a contradiction. Thus, we do not have stationary distribution.

Bounded Convergence Theorem

The following theorem appearing in analysis course provides a device to switch the order of limit and infinite sum effectively. We will see there is an application on the existence and uniqueness of stationary distribution.

Theorem. Suppose that
(i) the sequence of functions on positive integers, $a_n(k)$, $k \geq 1$, is *uniformly bounded*, i.e., there exists $K > 0$ such that

$$|a_n(k)| \leq K, \qquad \forall n, k \geq 1;$$

(ii) $\lim_{n\to\infty} a_n(k) = a(k)$ for any $k \in \mathbb{Z}_+$;
(iii) $\sum_{k=1}^{\infty} p_k = 1$ (or $< \infty$), and $p_k \geq 0$ for any $k \in \mathbb{Z}_+$.
Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_n(k) p_k = \sum_{k=1}^{\infty} a(k) p_k. \tag{1}$$

Bounded Convergence Theorem

Proof. Since $|a_n(k)| \le K$ and $\lim_{n\to\infty} a_n(k) = a(k)$,

$$|a(k)| \le K, \quad \forall k \in \mathbb{Z}_+.$$

For any $\varepsilon > 0$, there exists M such that

$$\sum_{k=M+1}^{\infty} p_k < \varepsilon/4K.$$

For any k, there exists $N(k) \in \mathbb{Z}_+$ such that for any $n > N(k)$,

$$|a_n(k) - a(k)| < \varepsilon/2.$$

Bounded Convergence Theorem

Set $N = \max_{1\le k\le M}\{N(k)\}$, then for any $n > N$,

$$
\begin{aligned}
& \left|\sum_{k=1}^{\infty} a_n(k)p_k - \sum_{k=1}^{\infty} a(k)p_k\right| \\
= & \left|\sum_{k=1}^{M} a_n(k)p_k + \sum_{k=M+1}^{\infty} a_n(k)p_k - \sum_{k=1}^{M} a(k)p_k - \sum_{k=M+1}^{\infty} a(k)p_k\right| \\
\le & \sum_{k=1}^{M} |a_n(k) - a(k)|p_k + \sum_{k=M+1}^{\infty} |a_n(k)|p_k + \sum_{k=M+1}^{\infty} |a(k)|p_k \\
\le & \sum_{k=1}^{M} (\varepsilon/2)p_k + \sum_{k=M+1}^{\infty} Kp_k + \sum_{k=M+1}^{\infty} Kp_k \\
< & \ \varepsilon/2 + (\varepsilon/4K)K + (\varepsilon/4K)K = \varepsilon.
\end{aligned}
$$

That implies the formula (1).